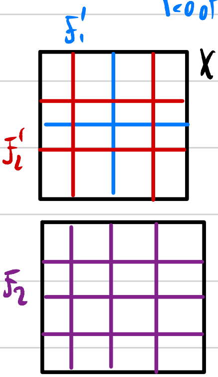
# Metric Spaces and Topology
## Lecture 25

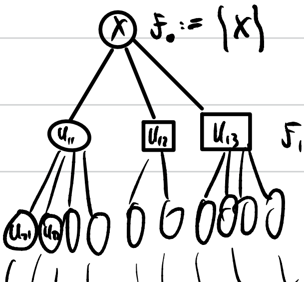
(3) $\Rightarrow$ (1). The proof is almost the same as for $[0,1]$.

Lemma. In a totally bdd metric space, there is a sequence $(\mathcal{F}_n)$, where each $\mathcal{F}_n$ is a finite open cover with sets of diameter $< \frac{1}{n}$ and s.t. $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$, i.e. each set in $\mathcal{F}_{n+1}$ is a subset of a set in $\mathcal{F}_n$.




Proof. Let $(F_n')$ be a sequence of finite $\frac{1}{n}$-nets, and let $\mathcal{F}_n' := \{B_{\frac{1}{2n}}(x) : x \in F_n'\}$, so it's a cover of $X$. Then let $\mathcal{F}_n$ be the set of all intersections of sets in $\mathcal{F}_1', \mathcal{F}_2', \dots, \mathcal{F}_n'$. $\square$

Now let $\mathcal{U}$ be an open cover of $X$ and suppose for the contrary that $\mathcal{U}$ doesn't have a finite subcover. Let $(\mathcal{F}_n)$ be the sequence of finite open covers of $X$ given by the lemma above. Thus, by Pigeonhole principle, there is a set $U_1 \in \mathcal{F}_1$ that doesn't admit a finite subcover with sets in $\mathcal{U}$.



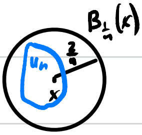

$\mathcal{F}_0 := \{X\}$

$\mathcal{F}_1 := \{U_{11}, U_{12}, U_{13}\}$

$\mathcal{F}_2 := \{U_{21}, U_{22}, U_{23}, \dots U_{2k}\}$

Similarly, $\exists\, U_2 \in \mathcal{F}_2$ that doesn't $U_2 \subseteq U_1$

have a finite subcover of $\mathcal{U}$, ... $\exists\, U_{n+1} \in \mathcal{F}_n$, $U_{n+1} \subseteq U_n$ that doesn't admit a finite subcover. Then $(\overline{U_n})$ is a decreasing sequence of closed sets of vanishing diameter, so $\exists\, x \in \bigcap_n \overline{U_n}$ by the completeness of $X$. But $\mathcal{U}$ cover $X$ so $\exists\, U \in \mathcal{U}$ s.t. $U \ni x$, hence $\exists\, n$ s.t. $B_{\frac{2}{n}}(x) \subseteq U$ and hence $\overline{U_n} \subseteq U$ because $\operatorname{diam}(U_n) < \frac{1}{n}$. Thus, $U_n$ admits a cover with a single set from $\mathcal{U}$, contradicting the very choice of $U_n$. $\square$




**Corollary.** In $\mathbb{R}^n$ with any of the equivalent metrics $d_p$, $1 \le p \le \infty$, compact sets are exactly the closed and bdd ones.

**Proof.** $\boxed{\text{HW}}$

## Examples of top. spaces violating (1) <=> (2).

**Order topology.** Given a totally ordered set $(X, <)$, i.e. $<$ is a total order on $X$: $\forall x, y, z \in X$,

(i) $x \not< x$

(ii) $x < y$ and $y < z \Rightarrow x < z$

(iii) $x < y$ or $y < x$ or $x = y$.

We define the order top on $X$ is generated by the intervals $(a,b) := \{x \in X : a < x < b\}$, for all $a < b$ in $X$.
Note that the in intersection of two intervals is again an interval, so the intervals form a basis for this topology.

**Example.** The usual Euclidean top on $\mathbb{R}$ coincides with the order top. with respect to the usual order.

**Example (ordinals).** Let $X := \omega_1$, i.e. the first unctbl ordinal. Ordinals are well-ordered sets by the order $\in$ and each ordinal $\alpha$ is equal to the ordinals less than it, indeed $\alpha = \{\beta : \beta \in \alpha\}$. Treating $\in$ as a strict well-order $<$ (in particular total order), consider the order top on $\omega_1$ with the addinal open set $\{0\}$, i.e. the basis is formed by intervals $(\alpha, \beta)$ and $\{0\}$, $\alpha < \beta$ in $\omega$.

**Obs 1.** $(\alpha, \beta] = (\alpha, \beta+1)$, so $(\alpha, \beta]$ is open.

**Cor 2.** The order top on $\omega_1$ is $1^{st}$ ctbl.
**Proof.** Indeed, by the observation above, for each $\beta \in \omega_1 \setminus \{0\}$,

the intervals $(\alpha, \beta]$, $\alpha < \beta$ form a neighbourhood basis because $\forall$ other interval $(\gamma, \delta) \ni \beta$, the interval $(\gamma, \beta] \subseteq (\gamma, \delta)$. But $\exists$ only ctbly many $\alpha < \beta$. $\square$

**Fact 3.** The supremum of a ctbl set $A$ of ctbl ordinals is still a ctbl ordinal.

**Proof.** Because $\bigcup A$ is a transitive set of ordinals, it's an ordinal. Using the fact that $\alpha \leq \beta \iff \alpha \subseteq \beta$ (where $\alpha \leq \beta$ means $\alpha \in \beta$ or $\alpha = \beta$), we see that $\bigcup A$ is the supremum (= least upper bound) for $A$. It remains to not that $\bigcup A$ is ctbl being a ctbl union of ctbl sets. $\square$

**Claim 4.** The order top. on $\omega_1$ is not compact; in fact it's not even Lindelöff: $\exists$ an open cover with no ctbl subcover.

**Proof.** Let $\mathcal{U}$ be the cover of $X$ consisting of intervals $(0, \alpha)$ and the set $\{0\}$, $\alpha \in \omega_1$.
This doesn't have a ctbl subcover $\{0\}, (0, \alpha_1), \dots, (0, \alpha_n), \dots$ hence then $\alpha := (\sup_{n \in \mathbb{N}} \alpha_n) + 1$ is still in $\omega_1$ by Fact 3, but $\alpha$ is not in any $(0, \alpha_n)$. $\square$

**Claim 5.** The order top. on $\omega_1$ is sequentially compact.

*Proof.* Let $(\alpha_n)$ be a sequence of ctbl ordinals. We may assume WLOG that $(\alpha_n)$ doesn't have a constant subsequence because such a subsequence would be convergent and we'd be done. Let $\beta_1 := \sup\{\alpha_n : n \in \mathbb{N}\}$, so $\beta_1 \in \omega_1$ by Fact 3. Only finitely many $\alpha_n$ (possibly none) can equal $\beta_1$, so removing these members we still get an infinite sequence, so the set $\{\alpha_n : n \in \mathbb{N}\} \setminus \{\beta_1\} \neq \emptyset$. Similarly, let $\beta_2 := \sup\{\alpha_n : n \in \mathbb{N}\} \setminus \{\beta_1\}$, so $\beta_2 \leq \beta_1$ and still $\{\alpha_n : n \in \mathbb{N}\} \setminus \{\beta_1, \beta_2\} \neq \emptyset$. Continuing, let $\beta_3 := \sup\{\alpha_n : n \in \mathbb{N}\} \setminus \{\beta_1, \beta_2\}$ and so on. The sequence $\beta_1 \geq \beta_2 \geq \beta_3 \geq \dots$ is a nonincreasing sequence of ordinals, so by well-orderedness, it must stabilize at some $k \in \mathbb{N}$, i.e. $\beta_k = \beta_{k+1} = \beta_{k+2} = \dots$ Thus, $\beta_k$ is not a max of $\{\alpha_n : n \in \mathbb{N}\} \setminus \{\beta_1, \dots, \beta_{k-1}\}$, i.e. the supremum is not achieved. It is now not hard to build a subsequence converging to $\beta := \beta_k$ using the fact that the intervals $(\gamma, \beta]$, $\gamma < \beta$, form a ctbl neighbourhood basis at $\beta$.

Indeed, enumerate $\{\gamma : \gamma < \beta\} = \{\gamma_\ell\}_{\ell \in \mathbb{N}}$, so

$\mathcal{B} := \{(\gamma_\ell, \beta]\}_{\ell \in \mathbb{N}}$ is a neighb. basis at $\beta$. We may assume WLOG that $B_\ell \supseteq B_{\ell+1}$ $\forall \ell$ by replacing $B_\ell$ with $\bigcap_{i=1}^{\ell} B_i$.
In other words, we may assume that $(\gamma_\ell)$ is increasing.
For $\ell = 1$, $\exists^\infty n$ s.t. $\alpha_n \in (\gamma_1, \beta]$ so take one and call it $\alpha_{n_1}$.
For $\ell = 2$, $\exists^\infty n$ s.t. $\alpha_n \in (\gamma_2, \beta]$ so take $n_2 > n_1$ with $\alpha_{n_2} \in (\gamma_2, \beta]$.
For $\ell = 3$, $\exists^\infty n$ s.t. $\alpha_n \in (\gamma_3, \beta]$ so take $n_3 > n_2$ with $\alpha_{n_3} \in (\gamma_3, \beta]$.
... We get a subsequence $(\alpha_{n_\ell})$ s.t. $\forall L \, \forall \ell \geq L$ $\alpha_{n_\ell} \in (\gamma_L, \beta]$, thus $\alpha_{n_\ell} \to \beta$ as $\ell \to \infty$. $\square$

To give an example of a compact space that is not sequentially compact, we need the following important theorem, which we will prove next time:

**Tychonoff's Theorem (AC).** Any product (possibly unctbl) of compact top spaces is compact (in the product top).

**Remark.** This theorem is equivalent to the Axiom of Choice. HW

**Example.** Consider the product space $10 := \{0, 1, \dots, 9\}^{[0,1]}$, i.e. all functions from $[0,1]$ to $10$. This is a compact space by Tychonoff's theorem but:

**Claim:** This space is not sequentially compact.

**Proof.** Let $(f_n)$ be the sequence of functions from $[0,1]$ to $10$, where $f_n(x) :=$ the $n^{th}$ digit after "$0.$" in the decimal rep. of $x$ (where we prefer the $0.**\!*999\dots$ notation to $0.**(*+1)000\dots$). For example $f_3(0.18703\dots) = 7$. This doesn't have a convergent subsequence. $\boxed{\text{HW}}$ $\square$